Senioru 6. IMO treniņa atrisinājumi.

Problem 9. Show that for infinitely many positive integers n there exist pairwise distinct positive integers $a_1, a_2, \ldots, a_n$ such that $a_1^2 a_2^2 \cdots a_n^2 - 4(a_1^2 + a_2^2 + \cdots + a_n^2)$ is the square of an integer.

Vietnam Olympiad, 2002

Solution. Letting $f_n(a_1, a_2, \ldots, a_n) = a_1^2 a_2^2 \cdots a_n^2 - 4(a_1^2 + a_2^2 + \cdots + a_n^2)$, the conclusion is a straightforward consequence of the following two facts:

(1) For infinitely many positive integers n there exist positive integers $a_1, a_2, \ldots, a_n$ such that $f_n(a_1, a_2, \ldots, a_n)$ is the square of an integer, and $a_1 < a_2$; and

(2) Given an integer $n \geq 3$, if $a_1, a_2, \ldots, a_n$ are positive integers such that $f_n(a_1, a_2, \ldots, a_n)$ is the square of an integer, and $a_1 < a_2 < \cdots < a_k$ for some index k, $1 < k < n$, then there exists an integer $a > a_k$ such that $f_n(a_1, a_2, \ldots, a_k, a, a_{k+2}, \ldots, a_n)$ is the square of an integer.

To prove **(1)**, notice that, if $a_1 < a_2$ are arbitrarily large positive integers, then $f_2(a_1, a_2)$ is an arbitrarily large positive integer congruent to 0 or 1 modulo 4. Subtraction of a suitable number of 4's then yields 0 or 1, each of which is a square. Letting that suitable number of 4's be $n - 2$, which is clearly arbitrarily large, $f_n(a_1, a_2, 1, \ldots, 1)$ is the square of an integer.

To prove **(2)**, write $f_n(a_1, a_2, \ldots, a_n) = b^2$ and notice that (a_{k+1}, b) solves the Pell equation

$$(a_1^2 \cdots a_k^2 a_{k+2}^2 \cdots a_n^2 - 4)x^2 - y^2 = 4(a_1^2 + \cdots + a_k^2 + a_{k+2}^2 + \cdots + a_n^2).$$

The latter has therefore infinitely many solutions in positive integers. In particular, it has a solution (a, c) such that $a > a_{k+1}$. This establishes **(2)** and concludes the prof.

Problem 16. Determine the largest value the expression $\sum_{1 \leq i < j \leq 4} (x_i + x_j)\sqrt{x_i x_j}$ may achieve, as x_1, x_2, x_3, x_4 run through the non-negative real numbers that add up to 1. Determine also the x_i at which the maximum is achieved.

The Editors

Solution. The required maximum is $3/4$ and is achieved if and only if the x_i are all equal to $1/4$. To prove this, use the binomial expansion of $(\sqrt{x_i} - \sqrt{x_j})^4$ to write

$$4(x_i + x_j)\sqrt{x_i x_j} = x_i^2 + 6x_i x_j + x_j^2 - (\sqrt{x_i} - \sqrt{x_j})^4,$$

then sum over $1 \leq i < j \leq 4$ and refer to the constraint $x_1 + x_2 + x_3 + x_4 = 1$, to get

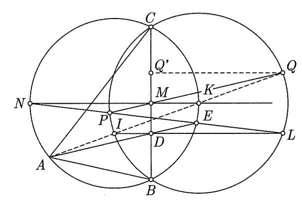
$$4 \sum_{1 \leq i < j \leq 4} (x_i + x_j)\sqrt{x_i x_j} = 3(x_1 + x_2 + x_3 + x_4)^2 - \sum_{1 \leq i < j \leq 4} (\sqrt{x_i} - \sqrt{x_j})^4$$
$$\leq 3(x_1 + x_2 + x_3 + x_4)^2 = 3;$$

clearly, equality holds if and only if the x_i are all equal, and the constraint forces them all equal to $1/4$.

Problem 17. Let ABC be an acute triangle such that $AB < AC$. Let I be the incentre of the triangle ABC, and let the incircle touch the side BC at D. The line AD crosses the circle ABC again at E. Let M be the midpoint of the side BC, and let N be the midpoint of the circular arc BAC. The line EN crosses the circular arc BIC at P. Show that the lines AD and MP are parallel.

Ukraine National Olympiad, 2016

Solution. The internal bisectrix AI and the perpendicular bisectrix MN of the side BC cross at the midpoint K of the arc BEC. It is a fact that the circle BIC is centred at K.



Let the lines ID and NPE cross at L. Notice that $\angle EAI = \angle EAK = \angle ENK = \angle ELI$ (since IL and KN are parallel), to infer that the quadrangle $AIEL$ is cyclic, so $DI \cdot DL = DA \cdot DE = DB \cdot DC$, showing that L lies on the circle BIC.

Since the angles KBN and KCN are both right, and the circle BIC is centred at K, the lines NB and NC are the tangents from N of this circle. It then follows that the line $NPEL$ is the P-simedian of the triangle BCP, so $\angle BPL = \angle CPM$.

Let now the line MP cross the circle BIC again at Q, to infer that the arcs BL and CQ of this circle have equal angular spans, so L and Q are reflexions of one another in the perpendicular bisectrix KMN of the chord BC.

Project Q orthogonally to Q' on BC and refer to standard notation in the triangle ABC: a, b, c denote the lengths of the sides BC, CA, AB, respectively, $s = (a + b + c)/2$ denotes its semiperimeter, r its inradius, and S its area. With reference to standard formulae, write $CQ' = BD = s - b$ and

$$QQ' = DL = \frac{DB \cdot DC}{DI} = \frac{(s-b)(s-c)}{r} = \frac{S}{s-a},$$

to infer that Q is the A-excentre of the triangle ABC, so it lies on the line AIK.

Finally, write $\angle PQA = \angle PQI = \angle PLI = \angle ELI = \angle EAI = \angle DAQ$, to conclude that the lines AD and MP are indeed parallel.

Remark. Another consequence of the above argument is that the A-excentre of the triangle ABC lies on the circle NPK.

Problem 18. Given an integer $n \geq 2$, colour red exactly n cells of an infinite sheet of grid paper. A rectangular grid array is called *special* if it contains at least two red opposite corner cells; single red cells and 1-row or 1-column grid arrays whose end-cells are both red are special. Given a configuration of exactly n red cells, let N be the largest number of red cells a special rectangular grid array may contain. Determine the least value N may take on over all possible configurations of exactly n red cells.

Based on Mathematical Olympiad Rioplatense, 2010, Level 2

Solution. The required minimum is $1 + \lceil (n+1)/5 \rceil$ and is achieved by the configuration described in the second block of the proof.

Given a configuration of exactly n red cells, we show that $N \geq (n+6)/5$. Consider the minimal rectangular grid array A containing the n red cells. By minimality, A contains some (not necessarily pairwise distinct) red cells a, b, c and d on the bottom row, the rightmost column, the top row, and the leftmost column, respectively; if, for instance, a is the lower-left corner cell, then the list reads a, b, c, a.

Letting $[xy]$ denote the (unique) rectangular grid array whose opposite corner cells are x and y, notice that the (not necessarily pairwise distinct) special rectangular grid arrays $[ab]$, $[bc]$, $[cd]$, $[da]$ and $[ac]$ cover A: the first two cover the part of A to the right of $[ac]$, and the next two cover the part of A to the left of $[ac]$.

Counting multiplicities, the cells a and c are both covered by three of these special rectangular grid arrays, the cells b and d are both covered by two, and all other red cells are covered by at least one. Letting $r_{[xy]}$ denote the number of red cells in $[xy]$, it follows that $r_{[ab]} + r_{[bc]} + r_{[cd]} + r_{[da]} + r_{[ac]} \geq 3 \cdot 2 + 2 \cdot 2 + (n-4) = n + 6$. Consequently, $N \geq (n+6)/5$.

We now describe a configuration of exactly n red cells where $N = 1 + \lceil (n+1)/5 \rceil$. Write $m = \lceil (n+1)/5 \rceil$, so $n = 5m - r$ for some positive integer $r \leq 5$, and $N = m + 1$.

Fix an integer $k > 2m$, let S be a $3k \times 3k$ grid square, and subdivide S into nine $k \times k$ grid subsquares.

Let S_{LL} be the lower-left corner $k \times k$ grid subsquare of S. Colour red the first m cells along the diagonal upward from the lower-right corner cell of S_{LL}.

The 'min' and 'max' in the next four paragraphs account for the first few cases where $m < r$. Had we assumed $n \geq 20$, it would then have followed that $m \geq r$, and 'min' and 'max' would have been superfluous.

Next, let S_{UL} be the upper-left corner $k \times k$ grid subsquare of S. Colour red the first $\min(m, 4m - r)$ cells along the diagonal upward from the lower-left corner cell of S_{UL}.

Let further S_{UR} be the upper-right corner $k \times k$ grid subsquare of S. Colour red the first $\min(m, 3m - r)$ cells along the diagonal downward from the upper-left corner cell of S_{UR}.

Complete the corner tour by letting S_{LR} be the lower-right corner $k \times k$ grid subsquare of S. Colour red the first $\min(m, 2m - r)$ cells along the diagonal downward from the upper-right corner cell of S_{LR}.

Finally, let S_C be the central $k \times k$ grid subsquare of S, and colour red $\max(0, m - r)$ cells of S_C; their exact location is irrelevant.

No other cell whatsoever is coloured red, and it is a routine exercise to check that exactly n cells of the grid paper have been coloured red. Notice that, for each pair of 'adjacent' corner $k \times k$ grid subsquares, S_{LL} and S_{UL}, S_{UL} and S_{UR}, S_{UR} and S_{LR}, and S_{LR} and S_{LL}, there are both horizontal and vertical grid lines separating the strings of red cells they contain.

To complete the argument, we show that, if x and y are red cells in this configuration, then $r_{[xy]} \leq m + 1$. This is clearly the case if x and y both lie in one of S_{LL}, S_{UL}, S_{UR}, S_{LR} or S_C, for each of these squares contains at most m red cells.

If x and y lie in 'adjacent' corner $k \times k$ subsquares of S, then the red cells in $[xy]$ come from those subsquares alone. In addition, the string of red cells in one of those subsquares has exactly one cell in $[xy]$, namely, x or y. Consequently, $r_{[xy]} \leq m + 1$. Incidentally, notice that equality holds if, for instance, x is the lower-right corner cell of S_{LL}, and y is any red cell in S_{UL}; since $n \geq 2$, there is at least one such.

If x and y lie in 'opposite' corner $k \times k$ subsquares of S, then they are the only red cells $[xy]$ contains from those subsquares. No red cell in the other two 'opposite' corner $k \times k$ subsquares of S lies in $[xy]$, and the other red cells in $[xy]$ all come from S_C which contains at most $m - 1$ such. Consequently, $r_{[xy]} \leq 2 + (m-1) = m + 1$.

Finally, if one of x, y lies in S_C, and the other lies in one of the corner $k \times k$ subsquares of S, then the latter cell is the only red cell in $[xy]$ outside S_C. Consequently, $r_{[xy]} \leq (m-1) + 1 = m < m + 1$. This ends the proof.