Senioru 5. IMO treniņa risinājumi.

Problem 11. Determine all functions f from the set of non-negative integers to itself such that

$$f(a+b) = f(a) + f(b) + f(c) + f(d),$$

whenever a, b, c, d, are non-negative integers satisfying $2ab = c^2 + d^2$.

RMM 2016 Shortlist, United Kingdom

Solution. (*Ilya Bogdanov*) The required functions are $f(n) = kn^2$, where k is a non-negative integer — these clearly satisfy the condition in the statement.

Conversely, let f be a function satisfying the condition in the statement. Setting $(a, b, c, d) = (n, n, n, n)$ in the functional relation yields $f(2n) = 4f(n)$ for all n. In particular, $f(0) = 0$ and $f(2) = 4k$, where $k = f(1)$.

Setting successively $(a, b, c, d) = (n^2, 1, n, n)$, $(a, b, c, d) = (n^2, 2, 2n, 0)$ and $(a, b, c, d) = (n^2 + 1, 1, n + 1, n - 1)$ in the functional relation yields

$$\begin{aligned} f(n^2+1) &= f(n^2) + k + 2f(n), \\ f(n^2+2) &= f(n^2) + 4k + f(2n) = f(n^2) + 4k + 4f(n), \\ f(n^2+2) &= f(n^2+1) + k + f(n+1) + f(n-1). \end{aligned}$$

Subtraction of the second relation above from the sum of the other two yields $f(n+1) = 2f(n) - f(n-1) + 2k$.

A straightforward induction on n now shows that $f(n) = kn^2$, for all non-negative n, and completes the proof.

Alternative Solution. As in the first solution, consider a function f satisfying the condition in the statement, and establish that $f(2n) = 4f(n)$, for all n; in particular, $f(0) = 0$.

We now show by induction on m that $f(mn) = m^2 f(n)$ for all n. By the preceding, this is clearly the case if $m = 0, 1, 2$. For the induction step, let $m > 2$.

Since $\sqrt{m} < m$, if m is a square, the conclusion follows by the induction hypothesis: $f(mn) = f(\sqrt{m}(\sqrt{m}\,n)) = mf(\sqrt{m}\,n) = m^2 f(n)$.

Otherwise, use Lagrange's four-square theorem to write $m = A + B$, where A and B are positive integers, each of which is a sum of two squares. Recall that if each factor of a product is a sum of two squares, then so is the product (use standard identities such as $(w^2 + x^2)(y^2 + z^2) = (wy + xz)^2 + (wz - xy)^2$ repeatedly), to write $2AB = (1^2 + 1^2)AB = C^2 + D^2$ for some non-negative integers C and D. Clearly, $A < m$ and $B < m$; $C \leq \sqrt{2AB} \leq (A + B)/\sqrt{2} = m/\sqrt{2} < m$, and similarly, $D < m$. Setting $a = An$, $b = Bn$, $c = Cn$ and $d = Dn$ in the functional relation and applying the induction hypothesis completes the induction step:

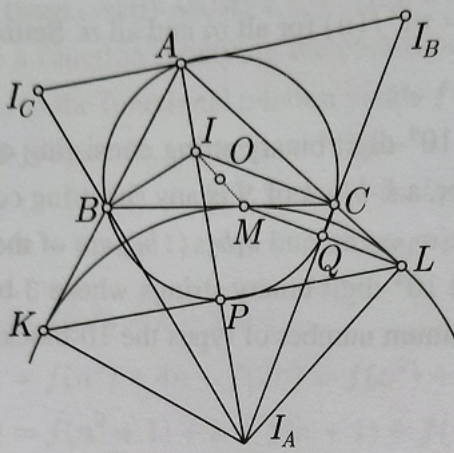
$$\begin{aligned} f(mn) &= f((A+B)n) = f(An) + f(Bn) + f(Cn) + f(Dn) \\ &= A^2 f(n) + B^2 f(n) + C^2 f(n) + D^2 f(n) = (A^2 + B^2 + C^2 + D^2) f(n) \\ &= (A^2 + B^2 + 2AB) f(n) = (A+B)^2 f(n) = m^2 f(n), \end{aligned}$$

for all non-negative integers n.

Consequently, $f(mn) = m^2 f(n)$ for all m and all n. Setting $n = 1$ and $k = f(1)$ concludes the proof.

Problem 13. Let ABC be a triangle, and let I and O be its incenter and circumcenter, respectively. The A-excircle touches the lines AB, AC, BC at K, L, M, respectively. Show that, if the midpoint of the segment KL lies on the circle ABC, then I, M, O are collinear.

Pavel Kozhevnikov, Russian Olympiad, 2005



Solution. Leaving the trivial case $AB = AC$ aside, we show that I, M, O all lie on the Euler line of the triangle formed by the three excenters I_A, I_B, I_C. Recall that the Euler line of a triangle is the line through the orthocenter, the center of the nine-point circle and the circumcenter of that triangle. Since I is the orthocenter of the triangle $I_AI_BI_C$, and O is the center of its nine-point circle, the line IO is indeed the Euler line of this triangle.

We now show that, if the midpoint of the segment KL lies on the circle ABC, then M is the circumcenter of the triangle $I_AI_BI_C$. The conclusion then follows by the preceding.

To prove that M is the circumcenter of the triangle $I_AI_BI_C$, we show that it lies on the perpendicular bisectrix of the segment I_AI_B ; similarly, it lies on the perpendicular bisectrix of the segment I_AI_C, so it is indeed the circumcenter of the triangle $I_AI_BI_C$.

Let P be the midpoint of the segment KL. Since $AK = AL$, the point P lies on the bisectrix AI of the angle BAC, so it is the midpoint of the circular arc BPC, and therefore lies on the perpendicular bisectrix of the segment BC ; and since B and C both lie on the circle on diameter II_A (the angles IBI_A and ICI_A are both right), it follows that P is the midpoint of the segment II_A.

Clearly, the line I_ACI_B is the perpendicular bisectrix of the segment LM, so it crosses the latter at its midpoint Q. Since PQ is a midline in the triangle KLM, it is parallel to KM, and since KM and BII_B are both perpendicular to I_ABI_C, it follows that PQ and BII_B are parallel. Recall that P is the midpoint of the segment II_A, to infer that PQ is a midline in the triangle II_AI_B, so Q is the midpoint of the segment I_AI_B. Consequently, M lies on the perpendicular bisectrix of the segment I_AI_B, as desired. This ends the proof.

Problem 14. Determine all positive integers n for which $n^{n+1} + n - 1$ is the sixth power of an integer.

Turkish Test for IMO, 2013

Solution. Clearly, $n = 1$ satisfies the required condition. We now proceed to rule out all integers $n > 1$.

If n is odd, $n \geq 3$, then $n^{n+1} + n - 1$ falls strictly between the squares of two consecutive integers,

$$\left(n^{(n+1)/2}\right)^2 < n^{n+1} + n - 1 < \left(n^{(n+1)/2} + 1\right)^2,$$

so it is not the square, and hence all the less the sixth power of an integer.

Similarly, if $n \equiv 2 \pmod 3$, then $n^{n+1}+n-1$ falls strictly between the cubes of two consecutive integers,

$$\left(n^{(n+1)/3}\right)^3 < n^{n+1}+n-1 < \left(n^{(n+1)/3}+1\right)^3,$$

so it is not the cube, and hence all the less the sixth power of an integer.

If $n \equiv 0 \pmod 3$, then $n^{n+1}+n-1 \equiv -1 \pmod 3$, so it is not the square, and hence all the less the sixth power of an integer.

Finally, to rule out the only case left, $n \equiv 4 \pmod 6$, notice that

$$n^{n+1}+n-1 \equiv (-1)^{n+1}-1-1 \equiv -3 \pmod{n+1}.$$

Since $n+1 \equiv 5 \pmod 6$, it has a prime divisor $p \equiv 2 \pmod 3$, $p > 3$, so -3 is not a quadratic residue modulo p. Consequently, $n^{n+1}+n-1$ is not a quadratic residue modulo p, and hence all the less the square of an integer, let alone the sixth power of one such.

Problem 15. Alice and Bob are given an integer $n \geq 2$ and a token to play the following game. Initially, Alice chooses an order of the numbers 1, 2, …, n, and writes them down in a row, in that order, on a sheet of paper. Next, Bob chooses one of these numbers and places the token on that number. Continuing, Alice moves the token on one of the neighbouring numbers, then Bob moves the token on one of the neighbouring numbers of the current position, and so on and so forth in turns. For each k in the range $1, 2, \ldots, n$, the token may be placed on number k at most k times; Bob's first move is counted. The player who can no longer move the token looses. Determine the values of n for which Alice has a winning strategy.

Mathematical Olympiad Rioplatense, 2010, Level 3

Solution. Alice has a winning strategy if and only if n is congruent to 0 or -1 modulo 4.

Looking upon placing the token on a number as a one unit drop of that number, a move is possible if and only if the current position of the token is not flanked by two zeroes. Notice that each player places the token on positions of like parity rank.

An arrangement $a_1, \ldots, a_n$ of an n-element collection of non-negative integers is *suitable* if there exist non-negative integers $x_0, x_1, \ldots, x_n$ such that

$$x_0 = x_n = 0, \quad \text{and} \quad a_k = x_{k-1} + x_k, \quad k = 1, \ldots, n.$$

Notice that, in a suitable arrangement, the a_k add up to twice the sum of the x_k, to infer that, for a collection of non-negative integers to have a suitable arrangement, it is necessary that the numbers in the collection add up to an even number. Otherwise, the collection has no suitable arrangement. In particular, if n is congruent to 1 or 2 modulo 4, there is no suitable arrangement of the collection 1, 2, …, n. If $n = 4m$, then

$$1 = 0 + 1, \quad 3 = 1 + 2, \quad 5 = 2 + 3, \quad \ldots, \quad 4m - 1 = (2m - 1) + 2m,$$
$$4m = 2m + 2m,$$
$$4m - 2 = 2m + (2m - 2), \quad \ldots, \quad 6 = 4 + 2, \quad 4 = 2 + 2, \quad 2 = 2 + 0$$

is a suitable arrangement of the collection $1, 2, \ldots, n$. Clearly, removal of $4m$ provides a suitable arrangement in case $n = 4m - 1$.

We are now in a position to show that Alice has a winning strategy if and only if the initial arrangement of numbers is suitable.

We first prove that, if the initial arrangement is suitable, then Alice wins. The idea is that Alice can counter any legal move of Bob, in a suitable arrangement, so as to make the new arrangement again suitable. Since the initial arrangement is suitable, and the game eventually comes to an end, Bob ultimately gets stuck and looses.

Let $a_1, \ldots, a_n$ be a suitable arrangement, and let Bob place the token on the k-th position. Since Bob's move is legal, $x_{k-1} + x_k = a_k > 0$. Without loss of generality, we may and will assume that $x_k > 0$, so $a_{k+1} = x_k + x_{k+1} \geq x_k > 0$, showing that the $(k + 1)$-st position is available for Alice to place the token on. After having done so, the new arrangement is again suitable, since a_k changed to $a'_k = x_{k-1} + (x_k - 1)$, a_{k+1} changed to $a'_{k+1} = (x_k - 1) + x_{k+1}$, and the other numbers are left unchanged. This establishes one implication.

Next, we show that, if the initial arrangement is not suitable, then Bob has a winning strategy. The idea is that this time Bob can achieve a suitable subarrangement just before one of Alice's moves, reset the whole game with this suitable subarrangement as initial arrangement, and thence set on to win by using the strategy described in the previous paragraph.

Let $a_1, \ldots, a_n$ be the initial arrangement; by assumption, it is not suitable. Let further $x_0 = 0$, and let $x_k = a_k - x_{k-1}$, $k = 1, \ldots, n$. Setting $a_{n+1} = 0$, it follows that $x_k > a_{k+1}$ for some positive integer $k \leq n$: Indeed, if $x_k \leq a_{k+1}$ for all indices $k \leq n$, then $x_1, x_2, \ldots, x_n$ are all non-negative; since the arrangement $a_1, \ldots, a_n$

is not suitable, $x_n > 0 = a_{n+1}$. Let m be the least index such that $x_m > a_{m+1}$, so $x_k \geq 0$ for all indices $k < m$.

Bob's first move consists in placing the token on the m-th position. After Bob's first move, a_m changes to $a_m - 1 = (x_{m-1} + x_m) - 1 = x_{m-1} + (x_m - 1) \geq x_{m-1} + a_{m+1}$. He may therefore place the token on the m-th position at least another $x_{m-1}+a_{m+1}$ times, regardless of Alice's placing the token on one of the neighbouring positions — at most a_{m+1} times on the $(m+1)$-st position, and at most a_{m-1} times on the $(m-1)$-st position (set $a_0 = 0$ if need be). Since $x_{m-1} + a_{m+1} \geq a_{m+1}$ (recall that $x_k \geq 0$ for all indices $k < m$), Bob is able to exhaust Alice's a_{m+1} allowed choices of the $(m+1)$-st position by keeping on placing the token on the m-th position. Thus, if still legal, her $(x_{m-1}+1)$-st move — call it move $(*)$ — consists in placing the token on the $(m-1)$-st position.

Just before move $(*)$, the first $m-1$ positions in the arrangement are $a_1, \ldots, a_{m-2}, x_{m-2}$, since $a_{m-1} = x_{m-2} + x_{m-1}$ dropped by x_{m-1}, and no move has been made on one of the first $m-2$ positions. At this stage, Bob can reset the whole game within the first $m-1$ positions: The $(m-1)$-element arrangement $a_1, \ldots, a_{m-2}, x_{m-2}$ is suitable, since $x_{m-2} \geq 0$, and he may reset $x_{m-1} = 0$ to fulfil the conditions in the definition. If still legal, move $(*)$ can now be looked upon as the first move in this suitable arrangement. Finally, by keeping on placing the token on one of these positions, Bob forces Alice to do so: Indeed, recall that just before resetting the game, his last move was on position m; if he now places the token on position $k \leq m-1$, then $k \leq m-2$, since k and m must share parity. This completes the argument and concludes the proof.