Senioru 4. IMO treniņa risinājumi.

Problem 5. Given an integer $n \geq 3$, determine the least value the sum $\sum_{i=1}^{n}(1/x_i - x_i)$ may achieve, as the x_i run through the positive real numbers subject to $\sum_{i=1}^{n} 1/(x_i + n - 1) = 1$. Also, determine the x_i at which this minimum is achieved.

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Solution. The required minimum is 0 and is achieved if and only if the x_i are all equal to 1. Let $x_1, \ldots, x_n$ be positive real numbers satisfying the condition in the statement. Let $y_i = x_i/(x_i + n - 1)$, $i = 1, 2, \ldots, n$, and notice that the y_i are positive real numbers that add up to 1. Express the x_i in terms of the y_i to get $x_i = (n-1)y_i/(1 - y_i)$, and write successively

$$\sum_{i=1}^{n} \frac{1}{x_i} = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1 - y_i}{y_i} = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{y_i} \sum_{j \neq i} y_j = \frac{1}{n-1} \sum_{i \neq j} \frac{y_j}{y_i} = \frac{1}{n-1} \sum_{i \neq j} \frac{y_i}{y_j}$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} y_i \sum_{j \neq i} \frac{1}{y_j} \geq \frac{1}{n-1} \sum_{i=1}^{n} y_i \cdot \frac{(n-1)^2}{\sum_{j \neq i} y_j} = \sum_{i=1}^{n} \frac{(n-1)y_i}{1 - y_i} = \sum_{i=1}^{n} x_i.$$

Equality clearly forces the y_i all equal to $1/n$, which is the case if and only if the x_i are all equal to 1.

Problem 6. Determine the largest integer N satisfying the following condition: for every cell labeling of a 5×5 array from 1 through 25 such that no two cells bear the same number, the numbers in some 2×2 square add up to at least N.

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Solution. The required maximum is $N = 45$, and is achieved, for instance, by the following extremal cell labeling:

25	5	24	6	23
11	4	12	3	13
22	7	21	8	20
14	2	15	1	16
19	9	18	10	17

In this cell labeling, every 2×2 square with an odd rank top row has sum 45, and every 2×2 square with an even rank top row has sum 44; the configuration is derived from the considerations below.

Write $n = 5$ and $m = \lceil n/2 \rceil = 3$, to show that, if n is odd, then for every injective cell labeling of an $n \times n$ array from 1 through n^2, the labels in some 2×2 square add up to at least $8m^2 - 11m + 6$ (= 45 in the case at hand).

Label rows downward and columns rightward, both from 1 through n, and let s_{ij} be the sum of all numbers labeling the cells in the cross formed by the i-th row and the j-th column. Formally, letting a_{ij} denote the label assigned to the cell on the i-th row and j-th column,

$$s_{ij} = \sum_{k=1}^{n} (a_{ik} + a_{kj}) - a_{ij}.$$

Use $\equiv$ to denote congruence modulo 2 and consider the sum

$$S = \sum_{i,j=1}^{m} s_{2i-1,2j-1} = \sum_{i \equiv j \equiv 1} s_{ij} = (2m-1) \sum_{i \equiv j \equiv 1} a_{ij} + m \sum_{i+j \equiv 1} a_{ij};$$

there are m^2 ordered pairs (i, j) such that $i \equiv j \equiv 1$, and $2m(n-m)$ ordered pairs (i, j) such that $i + j \equiv 1$.

The largest value the sum S may achieve is

$$S_0 = (2m-1) \sum_{k=0}^{m^2-1} (n^2 - k) + m \sum_{k=0}^{2m(n-m)-1} (n^2 - m^2 - k),$$

(which is clearly expressible in terms of m alone), so $s_{k\ell} \leq S_0/m^2$ for some odd indices k and ℓ; in the case at hand, $S_0 = 5 \times (25 + 24 + \cdots + 17) + 3 \times (16 + 15 + \cdots + 5) = 1323$, so $s_{k\ell} \leq 1323/9 = 147$.

The sum of the a_{ij} in the complement of C of the cross formed by the k-th row and ℓ-th column is therefore at least $(1+2+\cdots+n^2) - S_0/m^2 = n^2(n^2+1)/2 - S_0/m^2$; in the case at hand, at least $(1 + 2 + \cdots + 25) - 147 = 178$.

Finally, use the fact that n is odd, $n = 2m-1$, to tile C by $(m-1)^2$ squares 2×2 and infer that the numbers in one of these tiles add up to at least

$$\left\lceil \frac{n^2(n^2+1)}{2(m-1)^2} - \frac{S_0}{m^2(m-1)^2} \right\rceil = \left\lceil 8m^2 - 11m + 5 + \tfrac{1}{2} \right\rceil = 8m^2 - 11m + 6,$$

establishing the required lower bound.

Remark. In fact, if n is odd, $n = 2m-1$, then $N = 8m^2 - 11m + 6$, the extremal cell labeling being similar to the one displayed above: every 2×2 square with an odd rank top row has sum N, and every 2×2 square with an even rank top row has sum $N - 1$.

Problem 7. Let ABC be a non-rightangled triangle, let D, E and F be the feet of its altitudes from A, B and C, respectively, and let H be its orthocenter. Reflect E and F in the line AD to obtain the points E' and F', respectively. The lines BF' and

CE' cross at X, and the lines BE' and CF' cross at Y. Prove that the lines AX, BC and HY are concurrent.

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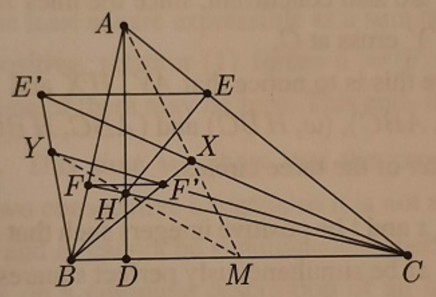
Solution. The angles AEH and AFH are both right, and the lines DE and DF are reflections of one another in the line ADH, so A, E, F', H, F, E' all lie on the circle ω on diameter AH.

Consider the hexagram $EHFF'XE'$: The lines EH and $F'X$ cross at B, the lines HF and XE' cross at C, and the lines FF' and EE' meet at the ideal point of the line BC (they are both parallel to the latter). Since E, H, F, F', E' all lie on ω, so does X, by the converse of Pascal's theorem. A similar argument, applied this time to the hexagram $FAEE'YF'$ shows that Y lies on ω as well.

The final argument hinges on Pascal's theorem applied to the cyclic hexagram $HEAXF'Y$: The lines HE and XF' cross at B, the lines EA and $F'Y$ cross at C, so the third pair of lines, AX and YH, cross on the line BC.



Alternative Solution. A rather lengthy, but less exotic and more down-to-earth argument shows that the lines AX and HY both pass through the midpoint M of the segment BC.

Before going any further, reduce the problem to showing that HY passes through M. To this end, notice that A, B, C, H form an orthocentric configuration: Each point is the orthocenter of the triangle determined by the other three; in particular, A is the orthocenter of the triangle HBC. This orthocentric correspondence preserves the roles of the points B, C, D, M — $B_{A,B,C,H} = B_{H,B,C,A}$ and the like —, and swops the roles of the points in the pairs (A, H), (E, F), (E', F') and (X, Y) — $A_{A,B,C,H} = H_{H,B,C,A}$ and the like. Consequently, $A_{A,B,C,H}X_{A,B,C,H}$ passes through $M_{A,B,C,H}$ if and only if $H_{H,B,C,A}Y_{H,B,C,A}$ passes through $M_{H,B,C,A}$, whence the reduction claim.

To prove that HY passes through M in the given configuration, recall the circle ω on diameter AH through E, E', F and F'. We will show that ω crosses the circle ABC again at Y. Then the angle AYH is right, so the line HY crosses the circle ABC again at the antipode of A which is the reflection of H across M. Consequently, M lies on the line HY.

Finally, we show that the circles ω and ABC cross again at Y. To this end, let the two circles cross again at Y' to write $\angle(BD, BY') = \angle(BC, BY') = \angle(AC, AY') = \angle(AE, AY') = \angle(F'E, F'Y') = \angle(F'D, F'Y')$, and infer that B, D, F' and Y' are concyclic. Therefore,

$$\angle(Y'B, Y'F') = \angle(DB, DF') = \angle(DC, DE) = \angle(AB, AC) = \angle(Y'B, Y'C),$$

showing that the points C, F' and Y' are collinear. Similarly, the points B, E' and Y' are collinear, so Y' and Y coincide. This completes the proof.

Remarks. Applied to the hexagram $AFHXF'Y$, Pascal's theorem shows that the lines AY, HX and BC are also concurrent, since the lines AF and XF' cross at B, and the lines FH and $F'Y$ cross at C.

Another way to prove this is to notice that AY, HX and BC are the radical axes of the pairs of circles (ω, ABC), (ω, HBC) and (ABC, HBC), respectively, so they concur at the radical center of the three circles.

Problem 8. Let x, y, z and t be positive integers such that $xy - zt = x + y = z + t$. Is it possible that xy and zt be simultaneously perfect squares?

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Solution. The answer is in the negative. Begin by noticing that $x + y = z + t$ is even. Otherwise, the summands on either side have opposite parities, so xy and zt are both even, and then so is $xy - zt = x + y$ which is a contradiction. Consequently, $x + y = z + t$ is even, and $s = \frac{1}{2}(x + y) = \frac{1}{2}(z + t)$ is a positive integer.

Suppose now, if possible, that $xy = a^2$ and $zt = c^2$ for some positive integers a and c. Consider the non-negative integers $b = \frac{1}{2}|x - y|$ and $d = \frac{1}{2}|z - t|$, and refer to the conditions in the statement to write

$$s^2 = a^2 + b^2 = c^2 + d^2, \tag{1}$$

and

$$2s = a^2 - c^2 = d^2 - b^2, \tag{2}$$

where the last equality in (2) follows from (1); in particular, (2) shows that d is positive.

In what follows only (1) and (2) will be dealt with, under the assumption that a, d and s are positive integers, while b and c are non-negative integers, at most one of which may be zero. Since (1) and (2) are both symmetric with respect to the simultaneous swops $a \leftrightarrow d$ and $b \leftrightarrow c$, we may and will assume that $b \geq c$, so b is positive. Therefore, $d^2 = b^2 + 2s > b^2 \geq c^2$, so

$$2d^2 > c^2 + d^2 = s^2. \tag{3}$$

On the other hand, since $d^2 - b^2$ is even by (2), the numbers b and d have the same parity, so $d - 2 \geq b > 0$, and therefore

$$2s = d^2 - b^2 \geq d^2 - (d-2)^2 = 4(d-1), \quad \text{i.e.,} \quad s + 2 \geq 2d. \tag{4}$$

Combine (3) and (4) to write $2s^2 < 4d^2 \leq (s+2)^2$ and infer that s is a positive integer strictly less than 5.

Finally, since 5^2 is the least square expressible as a sum of two non-zero squares, and a and d are both positive, relation (1) forces $b = c = 0$, contradicting the assumption that at most one of them may be zero. This ends the prof.

Alternative Solution. This time we show that the product $xyzt$ falls strictly between the squares of two consecutive integers, so it is not a perfect square. Consequently, the products xy and zt cannot be simultaneously perfect squares.

The argument hinges on a complete description of all quadruples (x, y, z, t) of positive integers satisfying

$$xy - zt = x + y = z + t. \tag{5}$$

As in the previous solution, notice that $s = \frac{1}{2}(x+y) = \frac{1}{2}(z+t)$, $p = \frac{1}{2}(x-y)$ and $q = \frac{1}{2}(z-t)$ are all integral; clearly, we may and will assume that p and q are both non-negative, that is, $x \geq y$ and $z \geq t$. Now write

$$2s = xy - zt = (s+p)(s-p) - (s+q)(s-q) = q^2 - p^2,$$

to infer that p and q have the same parity, and $p < q$. Next, consider the positive integers $u = \frac{1}{2}(q-p)$ and $v = \frac{1}{2}(q+p)$, to write $s = \frac{1}{2}(q^2 - p^2) = 2uv$ and get

$$\begin{aligned} x &= s + p = 2uv - u + v, & y &= s - p = 2uv + u - v, \\ z &= s + q = 2uv + u + v, & t &= s - q = 2uv - u - v; \end{aligned} \tag{6}$$

clearly, $v \geq u > 0$, and $(u, v) \neq (1, 1)$, since otherwise $t = 0$. Conversely, every such pair of integers yields via (6) a solution (x, y, z, t) to (5), where $x \geq y$ and $z \geq t$.

To show that the product $xyzt$ falls strictly between the squares of two consecutive integers, refer to formulae (6) to write

$$\begin{aligned} xyzt &= (2uv - u + v)(2uv + u - v)(2uv + u + v)(2uv - u - v) \\ &= \left(4u^2v^2 - (u - v)^2\right)\left(4u^2v^2 - (u + v)^2\right) = (4u^2v^2 - u^2 - v^2)^2 - 4u^2v^2 \\ &< (4u^2v^2 - u^2 - v^2)^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} (4u^2v^2 - u^2 - v^2 - 1)^2 &= (4u^2v^2 - u^2 - v^2)^2 - 2(4u^2v^2 - u^2 - v^2) + 1 \\ &= \left((4u^2v^2 - u^2 - v^2)^2 - 4u^2v^2\right) - (2u^2 - 1)(2v^2 - 1) + 2 \\ &= xyzt - (2u^2 - 1)(2v^2 - 1) + 2 \leq xyzt - 1 \cdot 3 + 2 \\ &= xyzt - 1 < xyzt, \end{aligned}$$

since $u \geq 1$ and $v \geq 2$. Consequently, $(4u^2v^2 - u^2 - v^2 - 1)^2 < xyzt < (4u^2v^2 - u^2 - v^2)^2$, as desired. This ends the proof.