Senioru 3. IMO treniņa risinājumi.

Problem 1. Given an integer $k \geq 2$, determine all positive integers $n_1, n_2, \ldots, n_k$ satisfying

$$n_2 \,|\, 2^{n_1} - 1, \quad n_3 \,|\, 2^{n_2} - 1, \quad \ldots, \quad n_k \,|\, 2^{n_{k-1}} - 1, \quad n_1 \,|\, 2^{n_k} - 1.$$

IMO 1985 Longlist, Romania

Solution. The required numbers are $n_1 = n_2 = \cdots = n_k = 1$.

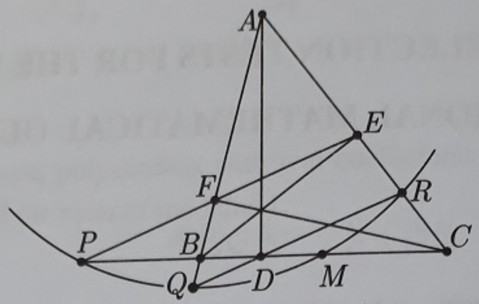
For every integer $m > 1$, let $p(m)$ denote the least prime divisor of m. We show that, if m and ℓ are integers greater than 1, and $m \,|\, 2^\ell - 1$, then $p(m) < p(\ell)$. Since $p(m)$ is odd, $p(m) \,|\, 2^{p(m)-1} - 1$, and since $p(m) \,|\, 2^\ell - 1$, it follows that $p(m) \,|\, 2^{\gcd(t, p(m)-1)} - 1$. Notice that $\gcd(\ell, p(m) - 1) > 1$, since $p(m) > 1$, to infer that ℓ has a prime divisor not exceeding $p(m) - 1$, and conclude thereby that $p(m) < p(\ell)$.

Suppose now, if possible, that $n_1 > 1$. Then $n_k > 1$, so $n_{k-1} > 1$, and so on and so forth all the way down to $n_2 > 1$. Hence $p(n_1) < p(n_2) < \cdots < p(n_k) < p(n_1)$ which is a contradiction. Consequently, $n_1 = 1$, so $n_2 = 1$, and so on and so forth all the way up to $n_k = 1$.

Problem 2. Let ABC be an acute triangle, and let D, E, F be the feet of the altitudes from A, B, C, respectively. The lines BC and EF cross at P, and the line through D and parallel to EF crosses the lines AC and AB at Q and R, respectively. Prove that the circle PQR passes through the midpoint of the side BC.

IMO 1997, Longlist

Solution. Let M be the midpoint of the side BC. If $AB = AC$, then P is the ideal point of the line BC, the points Q and R fall at C and B, respectively, and the circle PQR degenerates into the line BC on which M clearly lies.



Assume henceforth that $AB \neq AC$, say, $AB > AC$. It is clearly sufficient to show that

$$DM \cdot DP = DQ \cdot DR.$$

Since EF and QR are parallel, and B, C, E, F are concyclic (E and F both lie on the circle on diameter BC), so are B, C, Q, R. Hence $DB \cdot DC = DQ \cdot DR$, and it is therefore sufficient to show that $DB \cdot DC = DM \cdot DP$, i. e., $BM^2 = DM \cdot MP$, since $DB = BM + DM$, $DC = CM - DM = BM - DM$ and $DP = MP - DM$. Alternatively, but equivalently, $DP \cdot MP = MP^2 - BM^2$, since $DM = MP - DP$.

The points D, E, F, M are concyclic (they all lie on the nine-point circle of the triangle ABC), so $PD \cdot PM = PE \cdot PF$. The points B, C, E, F are also concyclic (recall that E and F both lie on the circle on diameter BC), so $PE \cdot PF = PB \cdot PC$. Consequently, $DP \cdot MP = PB \cdot PC = (BM + MP)(MP - CM) = (MP + BM)(MP - BM) = MP^2 - BM^2$, as desired.

Remarks. Acuteness of the triangle ABC was assumed to avoid degeneracy (such as rightness at B or C) or case analysis (the argument applies mutatis mutandis in case of obtuseness). Rightness at A causes no trouble: The line EF is the line through A and perpendicular to AM, i. e., the tangent of the circle ABC at A, and the argument goes along the same lines.

Keeping B and C fixed, while varying A in the plane so as to avoid the lines BC and the lines through B and C, respectively, and perpendicular to BC, the circle PQR passes through a fixed point – the midpoint of the segment BC.

Problem 3. Let a, b, c be positive integers such that $a < b < c$, and let f be the function of the positive integers into themselves, defined by $f(n) = n - a$ if $n > c$, and $f(n) = f(f(n+b))$ if $n \le c$. Determine the number of fixed points f may have.

Amer. Math. Monthly

Solution. The function f has exactly $b - a$ fixed points if a is divisible by $b - a$, and no fixed points at all otherwise. Clearly, f has no fixed point beyond c, so we focus on positive integers not exceeding c.

We first show recursively that $f(n) = f(n + b - a)$ for all positive integers $n \le c$. If $c - b < n \le c$, then $n + b > c$, so $f(n) = f(f(n+b)) = f(n+b-a)$; if $n \le c - b$, then $n + b - a < n + b \le c$, so $f(n+b-a) = f(n+2b-a)$; and if $c - 2b < n \le c - b$, then $c - b < n + b \le c$, so $f(n) = f(f(n+b)) = f(f(n+2b-a)) = f(n+b-a)$.

Assume now $f(n) = f(n+b-a)$, for $c - kb < n \le c - (k-1)b$, and consider a positive integer n, $c - (k+1)b < n \le c - kb$. Since $c - kb < n + b \le c - (k-1)b$, it follows that $f(n) = f(f(n+b)) = f(f(n+2b-a)) = f(n+b-a)$; and since $c - mb < 0$ for some positive integer m, the claim follows.

Let $n \le c$ and let $p = \lfloor (c-n)/(b-a) \rfloor$. Notice that $n + p(b-a) \le c$ and $n + (p+1)(b-a) > c$, to infer that $f(n) = f(n+b-a) = \cdots = f(n+p(b-a)) = f(n+(p+1)(b-a)) = n + (p+1)(b-a) + a$.

Thus, $f(n) = n$ if and only if $n \le c$ and $n + (p+1)(b-a) + a = n$, which is the case if and only if $(p+1)(b-a) = a$.

Consequently, f has no fixed points, unless a is divisible by $b - a$; in the latter case, f has a fixed point at n if and only if $\lfloor (c-n)/(b-a) \rfloor + 1 = a/(b-a)$, i.e., $c - a < n \le c - 2a + b$, in which case f has exactly $b - a$ fixed points.

Problem 4. Let m and n be positive integers, and let $A_1, \ldots, A_m$ be pairwise disjoint n-element sets of positive integers such that no member of A_i is divisible by one of A_{i+1}, whatever i (indices are reduced modulo m). Determine the largest number of ordered pairs (a, b), where a and b are members of distinct A_i's, and b is divisible by a.

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Solution. The required maximum is $\binom{m-1}{2}n^2$, and is achieved if, for instance

$$A_k = \{a^{(k-1)n+1}, a^{(k-1)n+2}, \ldots, a^{kn}\}, \quad k = 1, \ldots, m-1,$$

and

$$A_m = \{b, b^2, \dots, b^n\},$$

where a and b are relatively prime integers, both greater than 1.

For brevity, an ordered pair (a, b) satisfying the conditions in the statement will be called *suitable*. We show that the number of suitable pairs does not exceed $\binom{m-1}{2}n^2$.

For every m-tuple $(a_1, \dots, a_m)$, where a_k is a member of A_k, $k = 1, \dots, m$, let $k(a_1, \dots, a_m)$ be the number of suitable pairs (a_i, a_j) it contains.

We show by induction on m that $k(a_1, \dots, a_m) \le \binom{m-1}{2}$. Since there are exactly n^m m-tuples, and each suitable pair is contained in exactly n^{m-2} such, the conclusion follows.

The case $m = 3$ is easily dealt with. Let $m \ge 4$ and fix an m-tuple $(a_1, \dots, a_m)$, $a_k \in A_k$, $k = 1, \dots, m$; without loss of generality, we may and will assume that a_1 is the largest entry. The $(m-1)$-tuple $(a_1, \dots, a_{m-1})$ then satisfies the induction hypothesis: a_2 does not divide a_1, a_3 does not divide a_2, and so on and so forth, a_{m-1} does not divide a_{m-2}, and, by maximality, a_1 does not divide a_{m-1}.

We show that the number of suitable pairs containing a_m does not exceed $m - 2$. It then follows that $k(a_1, \dots, a_m) \le k(a_1, \dots, a_{m-1}) + m - 2 \le \binom{m-2}{2} + m - 2 = \binom{m-1}{2}$, as desired.

Notice that, for each k in the range 1 through $m - 1$, at most one of the pairs (a_k, a_m), (a_m, a_k) is suitable. If neither (a_k, a_m) nor (a_m, a_k) is suitable for some k, then the number of suitable pairs containing a_m is clearly at most $m - 2$. Otherwise, since the pairs (a_m, a_k) and (a_{k+1}, a_m), $k = 1, \dots, m-2$, are not simultaneously suitable, and the pair (a_1, a_m) is certainly not suitable, by maximality of a_1, it follows that the pairs (a_m, a_k), $k = 1, \dots, m-1$, are all suitable, contradicting the fact that a_m does not divide a_{m-1}. This ends the proof.