Senioru 2. IMO treniņa risinājumi.

Problem 9. *We consider the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ which are defined by $a_0 = b_0 = 2$ and $a_1 = b_1 = 14$ and by*

$$a_n = 14a_{n-1} + a_{n-2},$$
$$b_n = 6b_{n-1} - b_{n-2}$$

for $n \geq 2$.

Decide whether there are infinitely many integers which occur in both sequences.

(Gerhard Woeginger)

Answer. Yes.

Solution. Sequence (a_n) starts with values 2, 14, 198, 2786, 39202, 551614. Sequence (b_n) starts with values 2, 14, 82, 478, 2786, 16238, 94642, 551614. We therefore conjecture that $a_{2k+1} = b_{3k+1}$ holds for $k \geq 0$.

Shifting the recurrence yields

$$a_{n+2} - 14a_{n+1} - a_n = 0,$$
$$a_{n+1} - 14a_n - a_{n-1} = 0,$$
$$a_n - 14a_{n-1} - a_{n-2} = 0$$

for $n \geq 2$. Multiplying these recurrences by 1, 14 and -1, respectively, and taking the sum yields $a_{n+2} - 198a_n + a_{n-2} = 0$ and thus

$$a_{n+2} = 198a_n - a_{n-2}$$

for $n \geq 2$.

Shifting the recurrence of (b_n) yields

$$b_{n+3} - 6b_{n+2} + b_{n+1} = 0,$$
$$b_{n+2} - 6b_{n+1} + b_n = 0,$$
$$b_{n+1} - 6b_n + b_{n-1} = 0,$$
$$b_n - 6b_{n-1} + b_{n-2} = 0,$$
$$b_{n-1} - 6b_{n-2} + b_{n-3} = 0$$

for $n \geq 3$. Multiplying these recurrences by 1, 6, 35, 6 and 1, respectively, and taking the sum yields $b_{n+3} - 198b_n + b_{n-3} = 0$ and thus

$$b_{n+3} = 198b_n - b_{n-3}$$

for $n \geq 3$.

We see that the subsequences (a_{2k+1}) and (b_{3k+1}) have the same initial values $a_1 = b_1 = 14$ and $a_3 = b_4 = 2786$ and fulfil the same recurrence. This implies that $a_{2k+1} = b_{3k+1}$ for all $k \geq 0$.

From the given recurrence, it is obvious that the sequence (a_n) is strictly increasing. Thus we also get infinitely many values which occur in both sequences.

(Gerhard Woeginger) □

Problem 10. *Let ABC be a triangle and I its incenter. The circumcircle of ACI intersects the line BC a second time in the point X and the circumcircle of BCI intersects the line AC a second time in the point Y.*

Prove that the segments AY and BX are of equal length.

(Theresia Eisenkölbl)

Solution. We shall show that $AB = BX$ holds. Since $AB = AY$ then follows by the same argument, this completes the proof (see Figure 3).

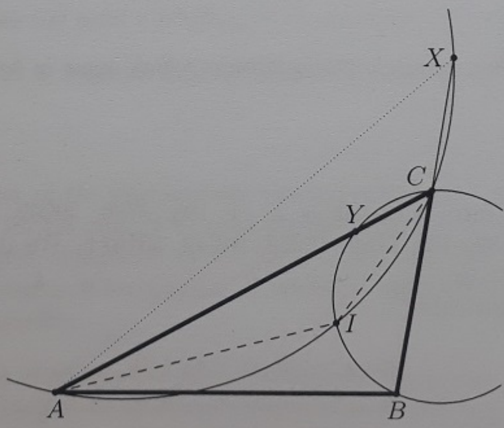

Figure 3: Problem 10

In this solution, we use oriented angles between lines (modulo 180°) with the notation $\angle PQR$. As usual the angles of the triangle ABC are denoted by $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$.

The inscribed angle theorem gives

$$\angle AXB = \angle AXC = \angle AIC = -\angle CIA = 180^\circ - \angle CIA = \angle IAC + \angle ACI = \frac{1}{2}(\alpha + \gamma).$$

This immediately implies

$$\angle BAX = -\angle AXB - \angle XBA = -\frac{1}{2}(\alpha + \gamma) - \beta = \frac{1}{2}(\alpha + \gamma).$$

Therefore, the triangle ABX is indeed isosceles, and we are done.

(Theresia Eisenkölbl) □

Problem 11. *Let $n \geq 2$ be an integer.*

Ariane and Bérénice play a game on the set of residue classes modulo n. In the beginning, the residue class 1 is written on a piece of paper. In each move, the player whose turn it is replaces the current residue class x with either $x+1$ or $2x$. The two players alternate with Ariane starting.

Ariane has won if the residue class 0 is reached during the game. Bérénice has won if she can permanently avoid this outcome.

For each value of n, determine which player has a winning strategy.

(Theresia Eisenkölbl)

Answer. Ariane wins for $n = 2$, 4 and 8, for all other $n \geq 2$ Bérénice wins.

Solution. We observe: If Ariane can win for a certain n, she will also win for all divisors of n, and conversely, if Bérénice can win for a certain n, she will also win for all multiples of n because a residue 0 modulo n is automatically a residue 0 for all divisors of n.

It remains to show that Ariane wins for $n = 8$ and Bérénice wins for $n = 16$ and n odd.

All congruences in this solution are modulo n.

- For $n = 8$, Ariane has to choose 2 in the first step. If Bérénice takes 4, Ariane can choose $8 \equiv 0$ and has won. If Bérénice takes 3, Ariane can choose 6. Now, Bérénice has to decide between 7 and $2 \cdot 6 = 12 \equiv 4$. But for both, Ariane can immediately choose $8 \equiv 0$.

- For $n = 16$, Bérénice chooses $2x$ for all numbers except 4 and 8. This clearly never gives the residue classes 0, 15 or 8, so that Ariane also cannot choose 0.

- For $n = 3$, Ariane has to choose 2 in the first step and then Bérénice chooses 1 again, which means that Bérénice wins.

- For odd $n > 3$, it is not possible to reach 0 with $2x$ from another residue class. So the only possible issue for Bérénice would be the situation that both her options are among n and $n-1$ such that she or Ariane choose 0. But this means that $x+1$ takes the residues 0 or -1, so $2x$ takes the residues -2 or -4 which are both different from 0 and -1, so this cannot happen and Bérénice can permanently avoid 0 being chosen.

(Theresia Eisenkölbl) □

Problem 12. *Find all pairs (a, b) of real numbers such that*

$$a \cdot \lfloor b \cdot n \rfloor = b \cdot \lfloor a \cdot n \rfloor$$

for all positive integers n.

(Walther Janous)

Answer. The solutions are all pairs (a, b) with $a = 0$ or $b = 0$ or $a = b$ or both a and b integers.

Solution. Let $a_0 = \lfloor a \rfloor$ and a_i be the binary digits of the fractional part of a such that $a = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{2^i}$ with $a_0 \in \mathbb{Z}$ and $a_i \in \{0, 1\}$ for $i \geq 1$. Similarly, let $b = b_0 + \sum_{i=1}^{\infty} \frac{b_i}{2^i}$ with $b_0 \in \mathbb{Z}$ and $b_i \in \{0, 1\}$ for $i \geq 1$. In the case of a non-unique binary expansion, we choose the expansion ending on infinitely many zeros.

Now choose $n = 2^k$ and $n = 2^{k-1}$ in the given equation. We get the equations

$$a\left(2^k b_0 + \sum_{i=1}^{k} b_i 2^{k-i}\right) = b\left(2^k a_0 + \sum_{i=1}^{k} a_i 2^{k-i}\right),$$
$$a\left(2^{k-1} b_0 + \sum_{i=1}^{k-1} b_i 2^{k-i-1}\right) = b\left(2^{k-1} a_0 + \sum_{i=1}^{k-1} a_i 2^{k-i-1}\right).$$

The first equation for $k = 0$ and the difference of the first equation and the doubled second equation for $k \geq 1$ yields

$$ab_k = ba_k \tag{1}$$

for $k \geq 0$.

Now, we consider three cases. If one or both of a and b are zero, then the original equation is clearly satisfied. If both fractional parts are zero, then both numbers are integers and again, the original equation is satisfied. So, finally, we consider the case that $a, b \neq 0$ and that there is a $k \geq 1$ with $a_k = 1$. The equation (1) shows that b_k cannot be zero, so we get $b_k = 1$ and thus from the same equation $a = b$. This clearly satisfies the original equation. (Of course, $b_k = 1$ leads to the same conclusion.) Therefore, the solutions are exactly the pairs listed in the answer.

(Theresia Eisenkölbl) □

Problem 13. *Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(2x + f(y)) = x + y + f(x)$$

for all $x, y \in \mathbb{R}$.

(Gerhard Kirchner)

Answer. The only solution is $f(x) = x$ for all $x \in \mathbb{R}$.

Solution. We choose x so that the arguments on both sides become equal, i.e. the equation $2x + f(y) = x$ is satisfied. For this value $x = -f(y)$, we get $f(-f(y)) = -f(y) + y + f(-f(y))$ and therefore $f(y) = y$ for all $y \in \mathbb{R}$. But this is clearly a solution, therefore, it is the only solution.

(Gerhard Kirchner) □

Problem 15. *In the country of Oddland, there are stamps with values 1 cent, 3 cent, 5 cent, etc., one type for each odd number. The rules of Oddland Postal Services stipulate the following: for any two distinct values, the number of stamps of the higher value on an envelope must never exceed the number of stamps of the lower value.*

In the country of Squareland, on the other hand, there are stamps with values 1 cent, 4 cent, 9 cent, etc., one type for each square number. Stamps can be combined in all possible ways in Squareland without additional rules.

Prove for every positive integer n: In Oddland and Squareland there are equally many ways to correctly place stamps of a total value of n cent on an envelope. Rearranging the stamps on an envelope makes no difference.

(Stephan Wagner)

Solution. We construct a bijection between possible combinations in Oddland and possible combinations in Squareland. Suppose we have a combination of Squareland stamps that sum to n cent, consisting of a_1 stamps of value 1 cent, a_2 stamps of value 4 cent, ..., a_M stamps of value M^2 cent, so that

$$n = \sum_{k=1}^{M} k^2 a_k.$$

Now we express k^2 as $\sum_{j=1}^{k}(2j-1)$ and interchange the order of summation, which yields

$$n = \sum_{k=1}^{M}\sum_{j=1}^{k}(2j-1)a_k = \sum_{j=1}^{M}(2j-1)\sum_{k=j}^{M} a_k.$$

This gives us a possible combination of Oddland stamps: By setting $b_j = \sum_{k=j}^{M} a_k$, we have

$$n = \sum_{j=1}^{M}(2j-1)b_j.$$

This can be interpreted as a collection of b_1 stamps of value 1 cent, b_2 stamps of value 3 cent, ..., b_M stamps of value $(2M-1)$ cent. We have $b_1 \geq b_2 \geq \cdots \geq b_M$ by definition, so this is a legal combination in Oddland.

Conversely, if a combination in Oddland is given by the values $b_1, b_2, \ldots, b_M$, we can use the identities $a_1 = b_1 - b_2$, $a_2 = b_2 - b_3$, ..., $a_{M-1} = b_{M-1} - b_M$, $a_M = b_M$ to recover the corresponding combination in Squareland. (Note that these values are nonnegative whenever $b_1 \geq b_2 \geq \cdots \geq b_M$.)

Since these two operations obviously are inverse to one another, we have found a bijection, which proves the statement.

(Stephan Wagner) □

Problem 17. *We are given an arbitrary acute-angled triangle ABC and its altitudes AD and BE where D and E denote their feet on sides BC and AC, respectively. Let furthermore F and G be two points on segments AD and BE, respectively, such that*

$$\frac{AF}{FD} = \frac{BG}{GE}.$$

The line through C and F intersects BE in point H and the line through C and G intersects AD in point I. Prove that the four points F, G, H and I are concyclic.

(Walther Janous)

Solution. The two right-angled triangles ADC and BEC are inversely similar to each other, see Figure 6. Here, the sides AD and BE correspond to each other.

But the condition

$$\frac{AF}{FD} = \frac{BG}{GE}$$

means: The two points F and G divide the two sides AD and BE, respectively, in equal ratios. Thus, the two oriented angles $\angle DFC$ and $\angle CGE$ are equal, which implies that the oriented angles $\angle IFH$ and $\angle IGH$ are equal modulo $180°$. Thus the inscribed angle theorem implies that the four points F, G, H and I are concyclic.

(Walther Janous) □

Remark. The solution only uses that $ABDE$ is inscribable.

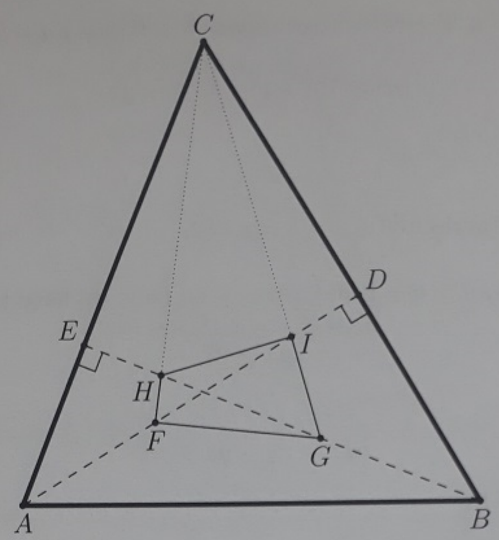

Figure 6: Problem 17

Problem 18. *Determine the smallest possible positive integer n with the following property: For all positive integers x, y and z with $x \mid y^3$ and $y \mid z^3$ and $z \mid x^3$ we also have $xyz \mid (x+y+z)^n$.*
(Gerhard J. Woeginger)

Answer. The smallest possible integer with that property is $n = 13$.

Solution. We note that we have $xyz \mid (x+y+z)^n$ if and only if for each prime p the inequality $v_p(xyz) \le v_p((x+y+z)^n)$ holds, where as usual $v_p(m)$ denotes the exponent of p in the prime factorization of m.

Let x, y and z be positive integers with $x \mid y^3$, $y \mid z^3$ and $z \mid x^3$. Let p be an arbitrary prime, and w.l.o.g. let the multiplicity of p be lowest in z, that is, $v_p(z) = \min\{v_p(x), v_p(y), v_p(z)\}$.

Then we have $v_p(x+y+z) \ge v_p(z)$, and from the divisibility constraints we get $v_p(x) \le 3v_p(y) \le 9v_p(z)$. It follows that

$$\begin{aligned} v_p(xyz) &= v_p(x) + v_p(y) + v_p(z) \\ &\le 9v_p(z) + 3v_p(z) + v_p(z) = 13v_p(z) \\ &\le 13v_p(x+y+z) = v_p((x+y+z)^{13}), \end{aligned} \tag{2}$$

which proves that for $n = 13$ the desired property is satisfied.

It remains to show that this is indeed the smallest possible integer with this property. For doing so, let n now be a number that has the desired property. By setting $(x, y, z) = (p^9, p^3, p^1)$ with an arbitrary prime p (in order to achieve that both inequalities in (2) become equalities), we get

$$\begin{aligned} 13 = v_p(p^{13}) &= v_p(p^9 \cdot p^3 \cdot p^1) = v_p(xyz) \\ &\le v_p((x+y+z)^n) = v_p((p^9 + p^3 + p^1)^n) = n \cdot v_p(p(p^8 + p^2 + 1)) = n, \end{aligned}$$

which yields $n \ge 13$.

(Birgit Vera Schmidt, Gerhard Woeginger) □