Senioru 1. IMO treniņa risinājumi.

1. We call a pupil a *singleton* if their best friend is in a different grade.

First give all singletons a destination as follows. Start by choosing any singleton A_1 and send them to Paris together with their best friend A_2. Suppose the destinations for pupils $A_{2m+1}, A_{2m+2}, \ldots, A_{2k}$, with $m \leq 0$ and $k \geq 1$ are chosen. If possible, first choose in the grade of A_{2k} a singleton A_{2k+1} who doesn't have a destination yet, and send them to the city different from the destination of A_{2k}, together with their best friend A_{2k+2}. If possible, then choose in the grade of A_{2m+1} a singleton A_{2m} who doesn't have a destination yet, and send them to the city different from the destination of A_{2m+1}, together with their best friend A_{2m-1}. Continue this process until pupils $A_{2m+1}, A_{2m+2}, \ldots, A_{2k}$ with $m \leq 0$ and $k \geq 1$ have destinations assigned to them and there are no singletons in the grades of A_{2m+1} and A_{2k} who don't have a destination yet. These pupils $A_{2m+1}, A_{2m+2}, \ldots, A_{2k}$ are by construction such that A_{2i-1} and A_{2i} are best friends and A_{2i} and A_{2i+1} have different destinations.

In every grade except that of A_{2m+1} and A_{2k} the same number of pupils is assigned to each destination. In the grades of A_{2m+1} and A_{2k} (which are different: otherwise no other singletons are in this grade, and all other pupils form pairs of best friends, making the number of pupils in that class even; contradiction!) one destination is assigned one more pupil than the other, and these grade no longer have any singletons without destination.

Repeat this process until every singleton has a destination. All remaining pupils now are in the same grade as their best friends, so every grade has an even number of pupils without destination and therefore an odd number of pupils with destination. So every grade contains a pupil on an end of exactly one of the sequences of pupils (which then must be the last time a pupil of this grade occurs in such a sequence). Therefore for each grade, one destination is assigned exactly one more pupil of that grade than the other.

Now consider any grade, and suppose that the number of singletons assigned to Paris is one more than that assigned to Rome. Give the pupils in this grade whose best friends are also in this grade a destination as follows. If the number of pairs of best friends is odd, first send one pair to Rome; otherwise do nothing. From the remaining even number of pairs of best friends, send half of them to Paris and half of them to Rome.

Now note that in the first step, if the number of pairs was odd, then Rome is assigned one more pupil than Paris. Therefore in any grade, the absolute difference of the number of students going to Paris and the number going to Rome is equal to 1. □

2. We assume that $a+b \le c+d$. As we can simultaneously interchange a, b and c, d without changing the problem, this assumption will not give any loss of generality. We also assume that $a \le b$ and $c \le d$; no generality is lost here as we can interchange a and b (resp. c and d) without changing the problem. Now we have $a^3 \le b^3$, so $\frac{a^2}{b} \le \frac{b^2}{a}$, and analogously we have $\frac{c^2}{d} \le \frac{d^2}{c}$. The given equation is now equivalent to

$$\frac{b^2}{a} \cdot \frac{d^2}{c} = (a+b)^4.$$

As $\frac{b^2}{a} \ge b$ and $\frac{d^2}{c} \ge d$, the left hand side is at least bd. We now show that $bd \le (a+b)^4$, from which follows that equality must hold everywhere.

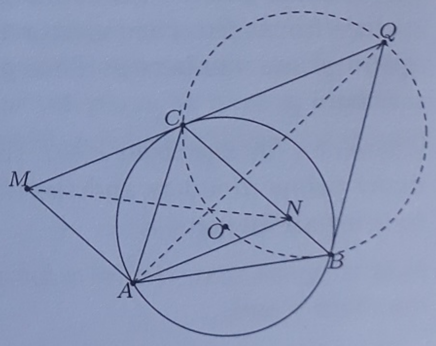
Since $a \le b$ we have $b \ge \frac{1}{2}(a+b)$, and analogously we have $d \ge \frac{1}{2}(c+d)$. Since $c+d \ge a+b$, it follows that $d \ge \frac{1}{2}(a+b)$. Moreover, from $a+b+c+d=1$ and $a+b \le c+d$ we deduce that $a+b \le \frac{1}{2}$, so

$$bd \le \tfrac{1}{4}(a+b)^2 \le (a+b)^2(a+b)^2 = (a+b)^4.$$

Therefore, each inequality used in the above must actually be an equality. Since we have used $\frac{b^2}{a} \ge b$ it follows that $a=b$, and analogously that $c=d$, and since we have used $a+b \le \frac{1}{2}$ it follows that $a+b=\frac{1}{2}$. Therefore $a=b=c=d=\frac{1}{4}$.

So the only possible solutions is $(a,b,c,d) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Substitution of this candidate solution gives $\frac{1}{16}$ on both sides of the equation, and therefore actually is a solution. □

3. Let T be the intersection of the circumcircle of $\triangle BOC$ with AQ. We show that T lies on NM. Write $\alpha = \angle BAC$. Then we have $\angle BOC = 2\alpha$ by the inscribed angle theorem. Since $|OB| = |OC|$ and $\angle OCQ = 90° = \angle OBQ$ by Thales's theorem, we have $\triangle OBQ \cong \triangle OCQ$. Hence we have $\angle COQ = \angle BOQ = \alpha$.



As $ANCM$ is a parallelogram, we see that $\angle AMC = 180° - \angle MCN = \angle QCN = \angle QCB = \angle QOB = \alpha$. Hence we have $\angle CNA = \angle AMC = \alpha$. Now we see that $\angle QTB = \angle QOB = \alpha = \angle CNA$, from which follows that $\angle ATB = 180° - \angle QTB = 180° - \angle CNA = \angle ANB$. Hence $ATNB$ is a cyclic quadrilateral. Also note that $ATCM$ is a cyclic quadrilateral since $\angle AMC = \alpha = \angle COQ = \angle CTQ = 180° - \angle ATC$.

As $ATCM$ is a cyclic quadrilateral, we have $\angle ATM = \angle ACM = 180° - \angle QCB - \angle BCA$, using supplementary angles at C. Recall that $\angle QCB = \angle QOB = \alpha = \angle CAB$, so using the sum of the angles in triangle ABC, we obtain $\angle ATM = 180° - \angle CAB - \angle BCA = \angle ABC$. Since $ATNB$ is a cyclic quadrilateral, we see that $\angle ABC = \angle ABN = 180° - \angle ATN$, so we have $\angle ATM = 180° - \angle ATN$. Therefore M, T, and N are collinear, as we have set out to prove. □

4. Suppose that f is a function satisfying both relations.

Substituting $x = p$ gives $p \mid \left(2f(p)\right)^{f(p)} - p$ and as p is prime, we have $p \mid 2f(p)$. So either $p = 2$ or $p \mid f(p)$.

Substituting $x = 0$ gives $p \mid \left(f(0) + f(p)\right)^{f(p)} - 0$ and as p is prime, we have $p \mid f(0) + f(p)$. Since $p \mid f(p)$ if $p \neq 2$, it follows that $p \mid f(0)$ if $p \neq 2$. So $f(0)$ is divisible by infinitely many prime numbers and therefore must be equal to 0. From $2 \mid f(0) + f(2)$ we now see that $2 \mid f(2)$, so $p \mid f(p)$ for all prime numbers p.

Now the second of the given relations translates to $f(x)^{f(p)} \equiv p \mod p$ for all integers x and prime numbers p. It follows that $p \mid f(x)$ if and only if $p \mid x$. Applying this observation to the case in which x is a prime number $q \neq p$, then we see that $f(q)$ is not divisible by any prime number $p \neq q$. As $f(q) > 0$, we deduce that $f(q)$ is a power of q. Fermat's Little Theorem states that for all prime numbers p and integers n we have $n^p \equiv n \mod p$, so we also have $n^{p^t} \equiv n \mod p$ for all non-negative integers t. As $f(p)$ is of the form p^t with t a non-negative integer for all prime numbers p, we deduce from the second given relation that $f(x) \equiv x \mod p$ for all integers x and prime numbers p. Thus $p \mid f(x) - x$ for all integers x and all prime numbers p.

Therefore, for any fixed $x \in \mathbb{Z}$ the integer $f(x) - x$ is divisible by infinitely many prime numbers and therefore equal to 0. Hence $f(x)$ must be equal to x for all integers x.

Now suppose that $f(x) = x$ for all integers x. Then $f(p) > 0$ for all prime numbers p and

$$\left(f(x) + f(p)\right)^{f(p)} - x = (x + p)^p - x \equiv x^p - x \equiv 0 \mod p$$

for all integers x and all prime numbers p by Fermat's Little Theorem.

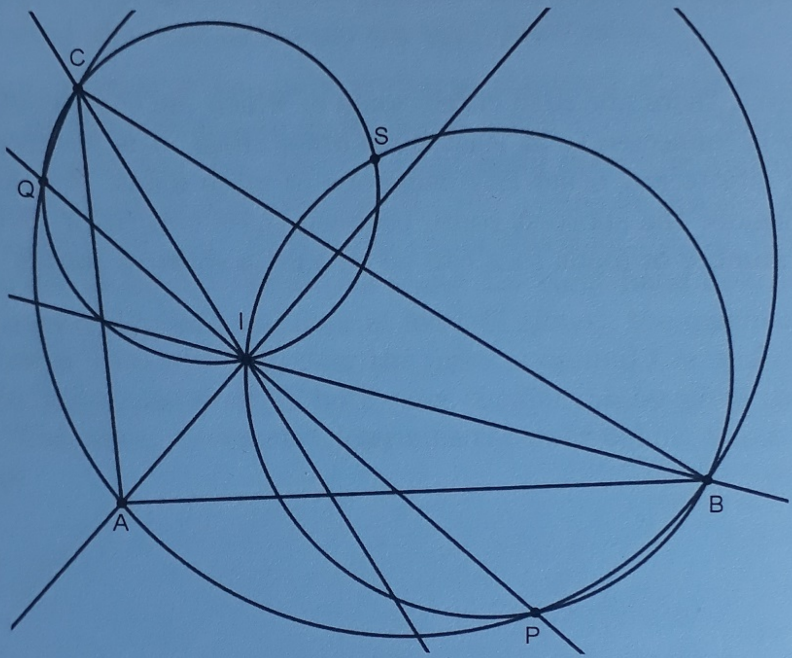
It follows that $f(x) = x$ is the unique function $\mathbb{Z} \to \mathbb{Z}$ satisfying both relations. □

2. Write $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$, and $\angle BCA = 2\gamma$. As the angles in triangle $\triangle AIB$ add up to $180°$, we have

$$\angle BIA = 180° - \angle IAB - \angle ABI = 180° - \alpha - \beta = 90° + \gamma,$$

hence

$$\angle BIP = \angle BIA - \angle AIP = 90° + \gamma - 90° = \gamma = \angle BCI.$$



Moreover, as $BPQC$ is a cyclic quadrilateral, we have

$$180^\circ - \angle BPI = 180^\circ - \angle BPQ = \angle BCQ = \angle BCI + \angle ICQ.$$

If we now consider the sum of the angles in triangle BPI, and use both previous results, we find

$$\angle PBI = 180^\circ - \angle BPI - \angle BIP = \angle BCI + \angle ICQ - \angle BCI = \angle ICQ.$$

Using the fact that $BSIP$ and $CQIS$ are cyclic quadrilaterals, we get

$$\angle PSI = \angle PBI = \angle ICQ = \angle ISQ,$$

hence SI is the angle bisector of angle PSQ. □

3.

1. We have $x^3 - 4x \geq y^3 > 0$, hence $x(x^2-4) > 0$. As x is positive, this yields $x^2 - 4 > 0$, hence $x^2 > 4$. That means that $x > 2$. Moreover, we have $x^3 - y^3 \geq 4x > 0$, hence $x > y$. The combination of these two results (which is allowed because x and y are both positive) gives $x^2 = x \cdot x > 2 \cdot y = 2y$.
2. We have $(x^4-4)^2 \geq 0$. Expanding yields $x^8 - 8x^4 + 16 \geq 0$. Because x is positive, we can multiply this with x without changing the inequality sign, hence we have $x^9 \geq 8x^5 - 16x$. The inequality in the assumption gives $x^5 - 2x \geq y^3$. If we combine this with the preceding inequality, we get $x^9 \geq 8(x^5 - 2x) \geq 8y^3$. It follows that $x^3 \geq 2y$. □

4. Such a k and a sequence do not exist. We prove this by contradiction, so suppose they do exist. Note that $a_n \mid a_{n+k}$ and $a_n \mid a_{n+k+1}$ for all $n \geq 1$. Using simple induction, it follows that $a_n \mid a_{n+\ell k}$ and $a_n \mid a_{n+\ell k+\ell}$ for all $\ell \geq 0$. We will prove by induction to m that $a_n \mid a_{n+mk+(m+1)}$ for all $0 \leq m \leq k-1$. For $m = k-1$, this follows from $a_n \mid a_{n+k\cdot k} = a_{n+(k-1)k+k}$. Now suppose that for a certain m with $1 \leq m \leq k-1$ we have that $a_n \mid a_{n+mk+(m+1)}$. We also know that $a_n \mid a_{n+mk+m}$. Therefore, as $a_{n+(m-1)k+m} = \gcd(a_{n+mk+m}, a_{n+mk+m+1})$, we also have $a_n \mid a_{n+(m-1)k+m}$. This finishes the induction argument. Substituting $m = 0$ yields $a_n \mid a_{n+1}$.

Because $a_n \mid a_{n+1}$, we also have $\gcd(a_n, a_{n+1}) = a_n$ for all n. Hence, $a_n = a_{n-k}$ for all $n \geq k+1$. Now we have $a_{n-k} \mid a_{n-k+1} \mid a_{n-k+2} \mid \cdots \mid a_n = a_{n-k}$. Because these are all positive integers, $a_{n-k}, a_{n-k+1}, \ldots, a_n$ must all be equal. This must be true for all $n \geq k+1$, hence the sequence is constant, which gives a contradiction. □

5. The greatest number of roads that can be in such a shortest route, is 1511. We first describe a country for which this number is attained. Divide the cities in 504 groups: two groups of five cities $(A_0, B_0, C_0, D_0, E_0)$ and $(A_{503}, B_{503}, C_{503}, D_{503}, E_{503})$, and groups of four cities (A_i, B_i, C_i, D_i) for $1 \le i \le 502$. For all i with $0 \le i \le 503$: connect A_i to B_i, connect A_i to C_i, connect B_i to D_i, and connect C_i to D_i. Moreover, connect E_0 to A_0, B_0, and C_0. Connect E_{503} to B_{503}, C_{503}, and D_{503}. For each $1 \le i \le 502$: connect B_i to C_i. Finally, for each $0 \le i \le 502$: connect D_i to A_{i+1}. Now every city is connect two exactly three other cities.

If we now want to travel from A_0 to D_{503}, then we have to travel through all A_i and D_i, because the only connections between the different groups are there, and each group is only connected to the previous and the next group. Moreover, for $0 \le i \le 503$, the city A_i is not connected to D_i, which causes the route within group i to go through either B_i or C_i. At least three of the four or five cities of each group must lie on the route. In total, this route visits at least $3 \cdot 504 = 1512$ cities and has at least 1511 roads.

We will now prove that the shortest route between two cities cannot have more than 1511 roads. Consider such a shortest route which visits the cities A_0, A_1, ..., A_k consecutively. Each of the cities A_i has one more neighbour besides A_{i-1} and A_{i+1}, which we will call B_i (note that the cities B_0, B_1, ..., B_k do not have to be distinct). Moreover, A_0 and A_k are connected to a third city, say $C_0 \neq B_0$ and $C_k \neq B_k$ respectively. If one of the cities B_i or C_i equals one of the cities A_j, then we could have found a shorter route by going directly from A_i to A_j (or vice versa), which would be a contradiction. Hence, the cities B_i and C_i are not equal to any of the cities A_j.

If one of the cities B_i is connected to four cities A_j, say B_i is connected to A_i, A_m, A_n, and A_p with $i < m < n < p$, then we could shorten the route by going from A_i to B_i and then to A_p. This makes us skip at least two cities of the original route (A_m and A_n) and in their place we only visit B_i, hence this new route is shorter, which would be a contradiction. Therefore, within the cities B_i with $3 \le i \le k-3$ there are at least $\frac{k-5}{3}$ distinct cities. For B_0 and C_0, the route can only be shortened if one of these cities equals B_i for certain $i \ge 3$. That would be a contradiction, hence B_0 and C_0 are two distinct cities, not equal to B_i for $i \ge 3$. In the same way, B_k and C_k are two distinct cities, not equal to B_i for $i \le k-3$. Altogether, we have $k+1$ cities on the route itself and at least $\frac{k-5}{3} + 2 + 2$ other cities. Therefore, $\frac{k-5}{3} + k + 5 \le 2018$, hence $4k - 5 + 15 \le 3 \cdot 2018 = 6054$, hence $4k \le 6044$, hence $k \le 1511$.

We conclude that the greatest number of roads occurring in a shortest route is 1511. □